LETTERS TO THE EDITOR

Conservation and capacitance

The reason why our discussion is not 'getting anywhere' (Parton C 1991 *Phys. Educ.* **26** 213) is because a long time ago we already got there (1989 *Phys. Educ.* **24** 256). In summary, when charge is set into motion, it constitutes an increasing current, which makes an increasing magnetic field, which requires energy. I have never maintained that the principal energy loss from a circuit in which a charged capacitor is connected to an uncharged capacitor arises from the spark—the spark is a fast switch, the connecting wires the 'aerial'.

I have attempted to clarify the issues involved but Chris Parton suggests that I have strayed far from the original topic. Yet the various mechanical analogies, such as liquids flowing between tanks, the coupling of gearwheels and pile driving, all fail to emulate the electromagnetic losses. But I get no thanks for the trouble I have taken in unearthing the relevant historical material concerning spark transmitters.

I hope that no readers were misled over my statement that welding transformers match the low impedance of a welding arc, typically less than 1 Ω, to a value suitable for connection to the mains. I agree that this is not in the sense of matching a loudspeaker to an amplifier or an aerial to a transmitter.

May I suggest that it is time to look at a different aspect of the original problem? An instructive area for further discussion would be to consider why the following statement, when applied to a charged capacitor and a resistor in a Dewar flask, is in principle erroneous: 'when the capacitor is connected to the resistor, the energy is dissipated as heat'.

Guy S M Moore
School of Physical Sciences, Hatfield Polytechnic, Herts AL10 9AB, UK

On mass and energy

The simple derivation of the equation $\Delta E = \Delta mc^2$ [1,2] in terms linear in v/c may be useful in elementary introductions of special relativity. It is based, however, on the energy of the photon, which belongs to the realm of quantum physics. This is an unpleasant feature, particularly in an elementary approach. 'It was Einstein's style forever to avoid the quantum theory if he could help it' [3]. There are other simple derivations, for example [4]. But a derivation along similar lines avoiding the concept of the photon is worth mentioning. Devised by Poincaré [5] and exploited by Einstein [6] in a somewhat modified version, it goes as follows (figure 1).

Take a hollow cylinder of mass M and length L floating in free space. Inside, on the left-hand side, an atom undergoes a transition from an excited state with energy E_e to the ground state with energy E_g and emits electromagnetic radiation to the right. According to Maxwell's electrodynamics the momentum of the radiation with energy $E_e - E_g$ amounts to $(E_e - E_g)/c$ and the cylinder takes up the recoil momentum $MV = (E_e - E_g)/c$ to the left. Until the radiation is absorbed after a time L/c by an atom on the right-hand side, the cylinder is shifted to the left by

$$D = \frac{VL}{c} = \frac{E_e - E_g}{Mc}\,\frac{L}{c} = \frac{(E_e - E_g)L}{Mc^2} \qquad (1)$$

The same final state can be reached by exchanging in the initial situation the atom in the excited state on the left with the atom in the ground state on the right. This cannot be achieved if the atoms in the two states have equal mass. The mass of the atom in the excited state m_e has to be greater than the mass of the atom in the ground state m_g. Exchanging (in a thought experiment) the atoms with constant velocity u endows the cylinder, according to momentum conservation, with velocity U: $(m_e - m_g)u = MU$. Since $U/u = D/L$ it is finally displaced to the left by

$$D = \frac{(m_e - m_g)L}{M}. \qquad (2)$$

Equating (1) and (2) leads to

$$E_e - E_g = (m_e - m_g)c^2.$$

An atom loses mass $m_e - m_g$ radiating energy $E_e - E_g$. The concept of the photon has not been exploited and even atoms can be replaced by macroscopic radiating bodies. The sketched derivation assumes 'only 19th-century physics' [2] and can be used at secondary school level. Only momentum conservation and the relation between energy and momentum of a train of electromagnetic waves are needed.

In fact, Einstein made up a cycle of the first change and the reverse of the second. Since in this case there are no internal changes in the cylinder he could claim that 'a body originally at rest cannot perform a translational motion if no other bodies act upon it' and entitled the article 'The principle of conservation of motion of the centre of gravity and the inertia of energy' [6].

At a higher level, e.g. second-year undergraduate, it is preferable to avoid the reference to electromagnetic radiation altogether and consider a two-step approach. First one studies the motion of a particle that does not change its internal state, e.g. an electron. After introducing four-vectors and the invariant proper time $d\tau = dt(1 - v^2/c^2)^{1/2} = dt/\gamma$ the four-momentum $(m\gamma c, m\gamma v) = (E/c, \boldsymbol{P})$ is obtained by multiplying the four-velocity with the particle mass [7]. Its spatial part for $v/c << 1$ tends as $\gamma \rightarrow 1$ to the Newtonian momentum mv in its temporal part, however, one has to develop γ: $E = mc^2 + \frac{1}{2}mv^2 + \ldots$. The constant term is introduced as the rest energy $E_0 = mc^2$, 'merely a convenient convention' [8].

In the second step reactions among interacting particles are studied. A particle of mass m_i emits two equal particles with equal mass m_1 and equal energy in opposite directions and becomes a particle of mass m_f. Energy conservation demands†

$$m_i c^2 = m_f c^2 + 2E_1. \qquad (4)$$

The equation can be put in the form

$$m_i - (m_f + 2m_1) = 2(E_1 - m_1 c^2).$$

The right-hand side represents the kinetic energy of the emitted particles and, according to the first step, the kinetic energy of an emitted particle is $E_1 - m_1 c^2 = m_1 c^2[1/(1 - v^2/c^2)^{1/2} - 1]$ 'which has a profound physical content' [8]. If the system of particles is not isolated and the emitted particles eventually give up their kinetic energy by collisions with other particles, the rest energy of the initial system is diminished. In such a context the rest energy can be considered to be a reservoir of energy, like other forms of internal energy. However, other conservation laws, particularly baryon and lepton conservation, restrict its change.

Both steps can be done simultaneously [8] in a somewhat more sophisticated derivation. This replaces the original Einstein procedure by the emission of two wave packets [9], which was the basis of Rohrlich's simplified approach [2]. The recent debate has shown that one should be rather cautious in teaching 'energy–mass equivalence' [10–14].

† In an exercise it can be shown that equation (2) is recovered in an inertial frame of reference moving with velocity u with respect to the initial particle in the direction of one of the emitted particles. The difference between the equation in the new frame and the equation in the old one gives (4) multiplied by $1/(1 - u^2/c^2)^{1/2} - 1$. It should be remarked that equation (1) is valid only if the particles are far away from each other and no longer interact.

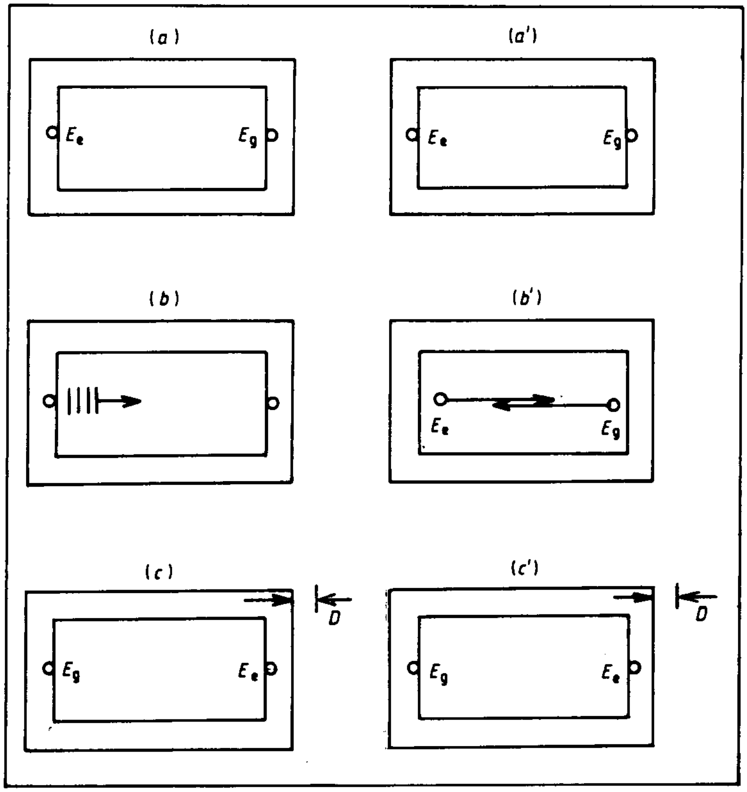

Figure 1. Initial situation with the atom in the excited state on the left and the atom in the ground state on the right (a), emission of radiation (b), final situation with the atom in the excited state on the right and the atom in the ground state on the left (c); initial situation (a'), exchange of atoms (b') and final situation (c'). In both changes the displacement of the cylinder to the left has to be the same.

J Strnad

Department of Physics and J Stefan Institute, University of Ljubljana, Ljubljana, Yugoslavia

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A simple derivation of $E=mc^2$

Rohrlich's paper [1], which V P Srivastava's letter summarizes (1991 *Phys. Educ.* **26** 214), states in the abstract, 'The equality $E=mc^2$ is derived in a fashion suitable for presentation in an elementary physics course for nonscience majors. It assumes only 19th-century physics and knowledge of the photon.' I wonder if Rohrlich had any reservations concerning its presentation to aspiring scientists, like I have, but additionally I am also apprehensive of its presentation to non-scientists.

Plenty of misconceptions concerning the supposed difficulties of studying special relativity exist, but all there is to know is held within two sentences:

1. The first postulate: 'It is impossible to measure or detect the unaccelerated translatory motion of a system through free space or through any ether-like medium assumed to pervade it.' [2]
2. The second postulate: 'The velocity of light in free space is the same for all inertial observers, independent of the relative velocity of the source of light and the observer.' [2]

To bypass this bedrock of relativity, and to run the risk of giving the impression that it is good enough to assume that light obeys the same Doppler shift principles as sound, is a highly questionable educational manoeuvre, in terms of both physics and history.

It is surely most important for persons capable of rational thought to be informed that, armed with these postulates and plenty of paper and time, all the formulae of special relativity could then be derived, including $E=mc^2$. This requires the application of reasoning to experiments which, in principle, could actually be carried out, such as the Michelson–Morley experiment, or considering the velocity of a bullet measured by an observer on a train and by an observer on the embankment, for each of whom the measured velocity of light travelling in any direction *in vacuo* is the same.

The full relativistic Doppler formula that applies to electromagnetic radiation, and which, by chance, I recently used in a cosmological application within the range of school mathematics (1991 *Phys. Educ.* **26** 211), cannot be deemed so complicated as to be beyond comprehension. I personally consider the lack of adequate discussion of this formula in many books on cosmology to be a serious scientific omission. If not all the facts are presented to us then this amounts to bias, and we are being taught 'science' as a matter of belief rather than one of developing our critical faculties.

Perhaps it should be admitted that many books on relativity, which promise by title to make the subject simple, are serious educational failures. A most useful exposition of special relativity is given in reference [3]—there are no short cuts to doing a good job and the mathematics presented there is at its simplest. Some of Einstein's original papers are conveniently studied in a 1 : 1 English translation in reference [4], including his proof of $E=mc^2$ in *less space* than Rohrlich's derivation *for the same experiment* in more general terms, using the full relativistic Doppler formula. I believe that Rohrlich's intriguing derivation certainly has a place in the context of dedicated examinations of the nature of physics, but it could be highly confusing if presented as a routine proof in introductory physics courses.

Guy S M Moore
School of Physical Sciences, Hatfield Polytechnic, Herts AL10 9AB, UK

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Put put rocket revisited

I was pleased to see that the rocket question, which I tried to answer in my earlier letter (*Phys. Educ.* **25** (1990) 304), seemed to stimulate several readers and enrich their course materials. The letter by Hinson (*Phys. Educ.* **26** (1991) 144) claimed that my conclusion, that it is better to throw out both balls at once, is incorrect. He said it makes no difference; the answer is the same. So I had to rethink the matter from a different perspective and see if he is indeed correct. I use no approximations, except that non-relativistic physics is correct.

I found that there is a subtle problem here of possibly comparing apples and oranges. Throwing out both identical balls is straightforward. Each gets a kinetic energy K_b and the ship recoils with a kinetic energy K_s, where $K_b+K_b+K_s\equiv E_0$ is a fixed and chosen amount of energy. No problem so far. If we instead throw out the balls one at a time, then we have a new parameter, the kinetic energy *chosen* to be given to the first ball, K_b'. We must then carefully adjust the throw of the second ball such that the two balls and the ship still appear to have a total kinetic energy, $K_b' + K_b'' + K_s'$, that equals E_0 (as seen in the *original* rest frame (ORF) of the rocket before anything was thrown out). Since momentum must be conserved, it is not obvious that this desired energy arrangement is still possible.

The first question is, 'Does it matter which value of K_b' is *chosen*, as far as the final K_s' is concerned, for fixed total kinetic energy in the first frame?' The answer is yes, as shown in my earlier letter. Having decided that, by doing a couple of specific rocket choices and using conservation of kinetic energy and momentum along x, we then have to ask, 'Is there a K_b' value that *maximizes* the speed of the recoiling rocket and, therefore, maximizes K_s'?' Solving for K_s' (or the speed, v_s') and then taking the derivative with respect to v_b' or K_b' for the first ball should give the value of K_b' that *maximizes* K_s'. The algebra gets very messy and I did not attempt this derivative. I found instead that if I just numerically chose K_b' to be the same value, K_b, as for simultaneous release, then even with conserving momentum, K_b'' and K_s' turn out to also have the same values as when both are thrown out at once. This follows from $K_b + K_b + K_s = K_b' + K_b'' + K_s'$, and from conservation of momentum as seen in the original inertial frame and in the second frame travelling with the rocket and second ball. By then choosing $K_b' = K_b \pm \varepsilon$ and following through the calculation of K_s', I found numerically that for small or large changes, ε, K_s' is *reduced.*

So Don Hinson is indeed right that it can make no difference, but *only if* one throws the first ball with just the right impulse. It must emerge from the moving ship with the same speed, as seen in the *initial* rest frame of the rocket, as both balls would have if thrown simultaneously. The remaining available energy, which is used to reverse the direction of the second ball, now travelling in the wrong direction, and used to further accelerate the rocket, turns out to be just enough to get it up to the same speed (relative to the *first* inertial frame) before releasing it from the rocket, where it of course appears to have a greater speed. Since both balls have the same speed (relative to the ORF) as when both were released simultaneoulsy, then clearly the ship must also have the same speed as when simultaneously released, since the *total* final kinetic energy available is assumed to be the same, as seen in the original inertial frame.

This time I numerically considered the even more exotic case of two very heavy balls, each 1000 times the mass of the ship, and a total energy available of $E_0/m_s = 10^6\ \mathrm{J\,kg^{-1}}$, where m_s is the mass of the 'ship' alone. These numbers give a final speed for the two exhaust chunks of only $0.7069\ \mathrm{m\,s^{-1}}$ and a ship recoil speed of $1413.8601\ \mathrm{m\,s^{-1}}$. To eject the first ball at this same low speed alone requires only $5 \times 10^{-2}\%$ of the total kinetic energy. Most of the total $10^6\ \mathrm{J\,kg^{-1}}$ goes into the second stage of acceleration of the ship, and into turning the second heavy ball around in direction and ejecting it, with the same speed as seen from the original frame. The rocket is then greatly speeded up in this second stage process, from the first stage speed of $0.7062\ \mathrm{m\,s^{-1}}$, obtained after the first ball was released, to the speed of $1413.8601\ \mathrm{m\,s^{-1}}$, as the second ball finishes being ejected at an original lab frame speed of $0.7069\ \mathrm{m\,s^{-1}}$. If we reduce the first ball ejection speed, as seen in the initial lab frame by a factor of 1/10, then the final ship speed *drops* only slightly, from $1413.8601\ \mathrm{m\,s^{-1}}$ to $1413.5741\ \mathrm{m\,s^{-1}}$. If we instead increase the speed of the first ball by a factor of 10, then the final rocket speed also *drops*, to $1384.8145\ \mathrm{m\,s^{-1}}$. To sort-of test that $K_b' = K_b$ really gives a maximum for K_s', rather than the maximum being possibly nearby, I tried increasing the first ball speed by only 12/10, and this also produced a small drop in final ship speed to $1413.8460\ \mathrm{m\,s^{-1}}$. It is very likely that $K_b' \equiv K_b$ is exactly the condition to maximize the recoil speed of the ship, with constrained total energy and two ejections. (This should be rigorously provable with a derivative as outlined above.)

It is not so easy to accelerate and eject, inside the rocket, the first ball so that it reaches a particular speed that is seen by an *outside* observer, not in the rocket. As the ball speeds up, the rocket speeds up in the opposite direction. I did not examine the question of what final speed is needed relative to the *rocket* as any particular ball is finally released (completely free from the rocket's influences on it). It should be easy to analyse this with the tools in my earlier letter. These numbers would be necessary to actually carry out the releases at the required speeds, by people in the rocket.

It requires a lot of care to release the first ball correctly. At first, it would seem easier to just throw them both out at once and then be assured that this gives the best recoil speed you can get. If there is a pile of balls to be thrown one at a time, as in the figure in my earlier letter, then each (I think) must be thrown, harder and harder in a carefully calculated way, relative to the ship at release. Each must reach the *same* final speed as seen in the *original* rest frame, though travelling behind each other in a row, in order to *match* the final boost speed that would be achieved by throwing all the balls out at this same speed, in one shot of acceleration, with the same energy release involved.

Of course, throwing all the balls out at *once* also requires that each be carefully accelerated up to speed relative to the ship at release, such that we have the desired final speed in the original lab frame. Since the ship is also speeding up as it recoils during this process, we have the same calculation and measurement problem on board as with consecutive releases. We have to transform to the rocket frame in order to get the ball speed that we can directly measure in the rocket, in preparing the balls for simultaneous release.

Suppose the astronaut used both arms, pushing full strength, to *simultaneously* release both *heavy* balls. For consecutive releases, can he push the first one out with one finger? If so, then *both* arms, pushing at almost full strength, would be needed to push the second ball out, for maximum ship speed. Calculate the final kinetic energy in each ball and in the recoiling ship after each stage of pushing. (Our bizarre rocket is a bit like a man shooting a gun where the bullet is a thousand times heavier than the man and gun combined.) The conditions for maximum recoil speed, obtained here for the rocket, should hold *regardless* of the mass of the balls compared with the ship.

My thanks to Mike Waldo for checking the calculations outlined above.

James D Edmonds Jr
Physics Department,
McNeese State University,
Lake Charles, LA 70609, USA